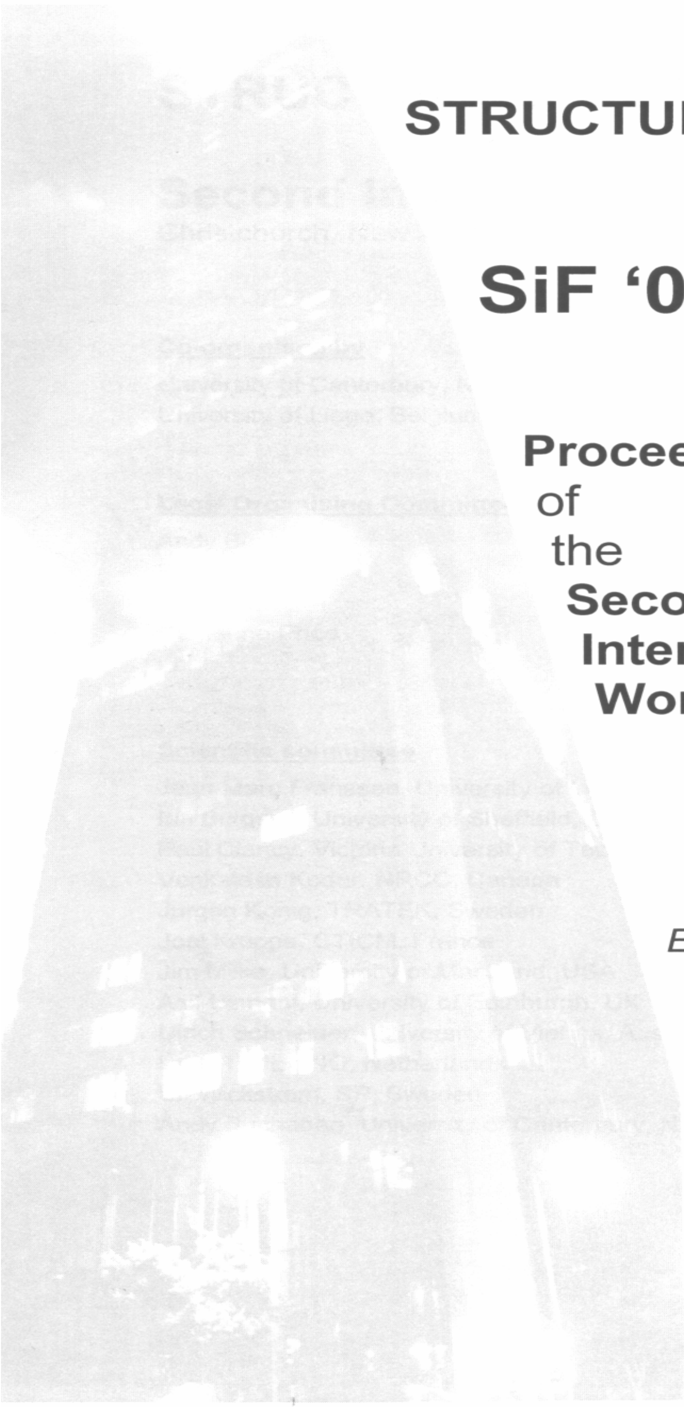

STRUCTURES IN FIRE

SiF '02

Proceedings of the Second International Workshop

Editor

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March 2002

This paper, in a slightly modified form, was presented at the Structures in Fire (SiF 02) Conference , University of Canterbury, Christchurch, New Zealand, March 2002

STABILITY OF LIGHTWEIGHT STRUCTURAL SANDWICH PANELS EXPOSED TO FIRE

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ABSTRACT

Sandwich panels comprising thin flat metal faces and a lightweight structural core are increasingly used as walls and ceilings in buildings where their long-span capabilities, high thermal insulation, clean design, rapid installation and low maintenance often make them the preferred choice of designers and building owners.

The fire performance of sandwich panels can be excellent if the correct core material is used and, importantly, if the metal facings are adequately restrained. For example, fire resistance in excess of 2 hours can be easily achieved using panels with sheet steel faces and a non-combustible rock wool core.

Where sandwich panels are used in cold stores there is the potential problem of cold-bridging between the facings wherever there is a metallic through-fixing, and this has led to designs which work well in normal conditions but allow panels to collapse very early when exposed to fire because the facings are not tied back to the supporting structure. Such collapse is a fire hazard to fire-fighters as proven in the 1993 fire in the Sun Valley poultry factory in Hereford, United Kingdom in which two firemen lost their lives.

The paper describes what can happen if panel facings are not mechanically restrained with steel fastenings. It then introduces a fire safety engineering method for assessing the stability of ceiling sandwich panels exposed to fire. The method assumes that the ends of panels are restrained and the panel behaves as a catenary after delamination. The paper quantifies the variation of catenary force as fire develops and takes account of the initial beneficial sag which is present at the time of delamination. The method is currently being considered in the work of European committee CEN TC 127 on the development of rules for extended applications for construction products.

KEYWORDS: *fire engineering, fire resistance, sandwich panel, stability, collapse, fire scenarios, ceilings, walls, cold stores*

INTRODUCTION

Sandwich panels are being increasingly used in single-storey and multi-storey buildings because they are lightweight, energy efficient, aesthetically attractive and can be easily handled and erected. When constructed with non-combustible structural rock wool cores, panels have good airborne sound insulation and levels of fire resistance which can exceed two hours.

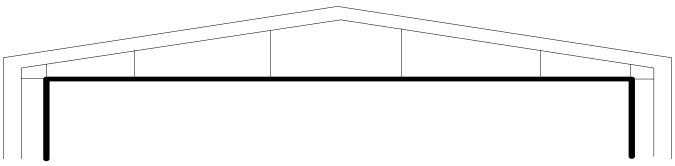
Many panels employ combustible cores of foamed plastic e.g. polyurethane, polyisocyanurate and polystyrene, which, in a fire, can delaminate and produce large amounts of heat, smoke and toxic gases which can be a hazard to life, property, business continuity and the environment.

Most sandwich panels in the UK have sheet steel facings and are bonded to the core using a thermosetting adhesive such as polyurethane. Small scale tests by the UK Fire Research Station, Building Research Establishment have shown that delamination temperatures are likely to be in the range 130-350 ^{o}C. This means that panels can delaminate and collapse before flashover unless the panel facings are adequately restrained.

Sandwich panels used as external wall and roof cladding are usually attached to a supporting structure which prevents both panel facings from falling down in a fire. However, when used as ceilings and free-standing internal walls (as in some cold stores), bonded sandwich panels can collapse if the facings are not adequately restrained. Configurations of panel use are shown in Figure 1

a)

b)



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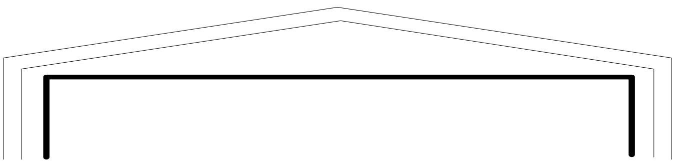

Figure 1 Different ways of using sandwich panels in a framed building

In Figure 1a) the panels (shown by the thick black line) are attached to the outside of the framework and this application is structurally safe provided both metal faces are mechanically attached to the frame. If only the inner metal face is attached to the frame it is possible for the outer face to delaminate in fire and act as a missile in strong winds. In Figure 1b) the panels are suspended from the frame. This application is structurally safe provided both facings of the wall panel ends are supported from the roof structure and, more importantly, both faces of the ceiling panels are adequately restrained vertically and horizontally in the fire. The free-standing, unrestrained configuration in Figure 1c) is, in most cases, unacceptable if collapse of the panels is to be prevented.

If a fire resistance test on a representative specimen has been made successfully it is unlikely that collapse in the building context will occur for an equivalent fire severity, similar size of panel and adequate panel-end restraint. However, not all sandwich panels are tested for fire resistance and an assessment then needs to be made for panel stability, especially if the panel in use is much larger than its fire-tested counterpart.

This paper deals solely with theoretical aspects of structural behaviour in fire. Information on other aspects such as fire load, fire scenarios, fire testing and a check list for fire safe design is available [1-3] while some preferred panel attachment methods are given in [4]. Following a number of damaging fires associated with plastic foam cored sandwich panels in the food industry two codes of practice have been published in the UK [5, 6]. A book on sandwich panels [7] has recently been published which has useful practical guidance on ways of designing panels to resist fire. The author has made an in-depth study [8] of the problems of making a risk assessment which properly takes account of the problems associated with panels having combustible cores, and he contends that present official UK technical guidance in the government's Approved Document B remains unsatisfactory. Some inadequacies in ad-hoc fire tests for combustible-cored sandwich panels are reported elsewhere [9].

FREE-STANDING INTERNAL WALLS

The stability of free-standing sandwich panels forming a wall is achieved by attaching both facings of the panel at the top, e.g. to a roof beam, which has the required fire resistance. In a fire the panel loses its flexural strength when the facing delaminates from the core and the panel facings then become suspended from the top.

Adequate suspension is achieved if:

a) the fastenings at the top of the fire exposed face carry the dead load of that facing,
b) the fastenings at the top of the unexposed face carry the dead load of that facing and the core, and
c) the top support member is capable of carrying the panel dead load.

Fire is an accidental limit state and, because the simultaneous occurrence of fire and snow is unlikely in most countries, the reserve of strength needed to carry the snow load can be utilised to carry the panel dead load in the fire condition so that the strength of the roof structure does not have to be increased.

CEILINGS

Structural sandwich panels rely on an adhesive layer between the flat metal faces and the core material for their flexural strength. Most adhesives used in proprietary panels delaminate at quite low temperatures, typically in the range 130 to 350°C as mentioned earlier. These temperatures are reached in less than 5 minutes in the ISO 834 fire resistance test exposure and well before flashover in a real fire. If panels simply rest on supports with no horizontal restraint, the panels will, on delamination, sag and slip off the supports. Since panels can be more than 1 m wide and 12 m long a collapsing ceiling panel is a substantial potential missile threat to occupants making their escape or fire-fighters performing their search, rescue and firefighting duties. To prevent collapse the ends of the panel faces must be fastened to the supporting structure and horizontally restrained so that they act as catenaries (cable-like) structures. This can be done without forming a thermal bridge between the upper and lower facings, and a suitable detail is shown in Figure 1.

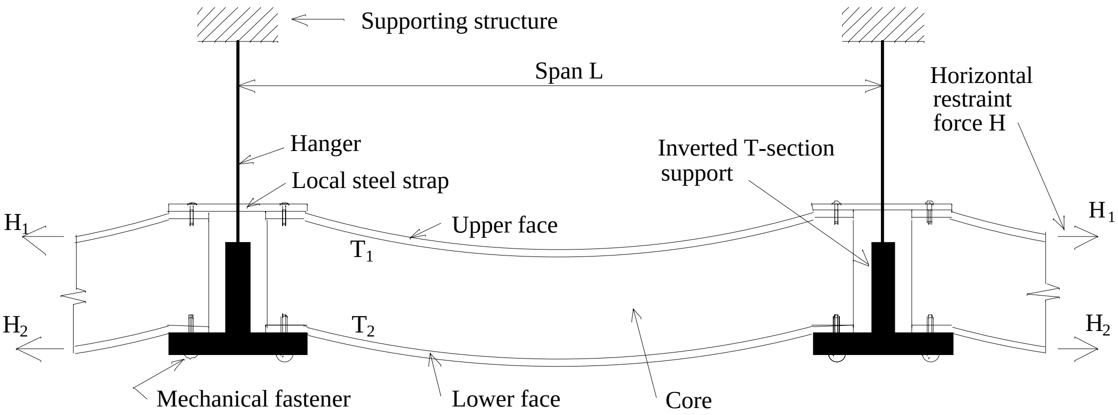

Figure 1 A preferred way of supporting and restraining ceiling panels

The catenary forces, H_1 and H_2, can be large and some attempt should be made to calculate them to ensure that the panel end fastenings do not fail. For a simply supported panel of span L, the horizontal force H needed to support the catenary is given by equation (1). This well-known equation is derived in Annex A

$$H = wL^2/8D \qquad (1)$$

where w = uniformly distributed load per unit length

Deflection D may be caused by thermal expansion of the facing and by inward displacement of the panel ends due to in-plane flexibility of the panel assembly. Both effects are beneficial.

Before equation (1) can be used, D must be calculated or estimated. This can be done if the temperature of the facing and/or inward end movement is known or can be estimated, and it is assumed in the following equations that the facings hang in the shape of a circular arc. It is shown in Annex B that, due to temperature rise alone, the deflection of a uniformly heated flexible member is (see Eq'n B9):

$$D = L(0.375\alpha T)^{1/2} \qquad (2)$$

where α = coefficient of thermal expansion, and
T = temperature rise

Similarly, due to inward end movement alone, for example by movement of the supports (see Eq'n B8):

$$D = (0.375Lp)^{1/2} \qquad (3)$$

where p = relative inward movement of panel ends

Using equations (1) and (2) example calculations have been made for the catenary force and mid-span deflection as a function of temperature rise for one steel facing which is 1200 mm wide by 0.5 mm thick. Curves are plotted for panel spans of 4.5m and 10m thus enabling a comparison of a tested construction (4.5m span) and a practical span in use taken as 10m. The results are shown in Figure 2. The higher the failure temperature of the adhesive, the smaller the sag, the lower the catenary force and the easier it

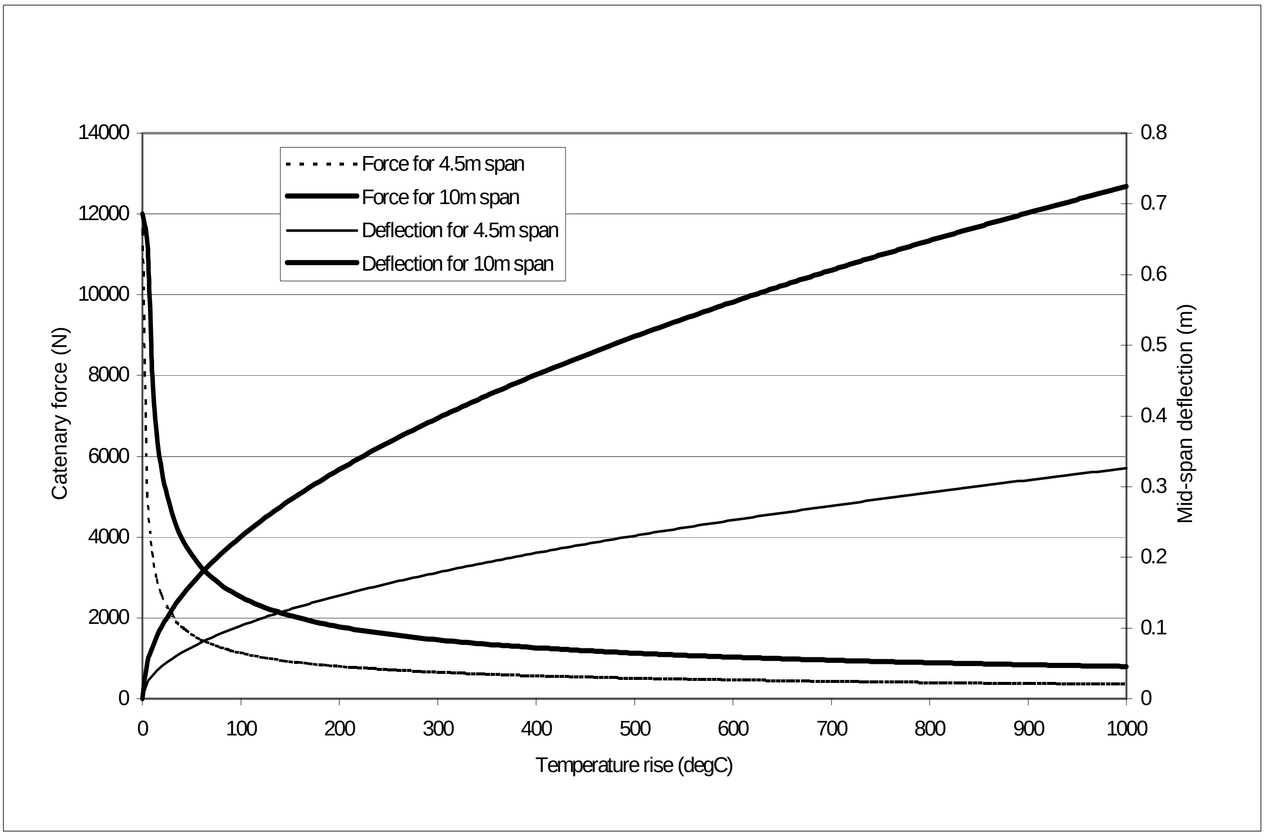

Figure 2 Variation of catenary force and deflection with temperature

is to design the panel end fastenings. The same kind of calculations can be made for the effect of panel inward end movement.

The catenary force will, in theory, cause an elastic extension in the facing which will lead to a small increase in mid-span deflection and a further reduction in catenary force. Thus, in a rigorous theory, H has to be derived by a process of iteration. However, the author believes that this effect (elastic extension) can usually be ignored in practice, as the following example shows:

$E = \sigma/\varepsilon$ where $\sigma = H/A$ and $\varepsilon = \delta/L$

Where E = elastic modulus (typically 200,000N/mm^2 for steel)
σ = tensile stress
ε = strain
H = catenary force
A = cross section area
δ = axial extension
L = length of member

Therefore $\delta = \dfrac{LH}{EA}$ (4)

For a span of 10m the catenary force H, from Figure 2, is 2520N for a temperature rise of 100^{o}C. Hence, from Eq'n (4):

$$\delta = \frac{10 \times 1000 \times 2520}{200{,}000 \times 1.2 \times 1000 \times 0.5}$$

$$\delta = 0.21mm$$

It is clear that an extension of 0.21mm on a 10m length will have a negligible effect on the mid-span deflection and catenary force. Thus elastic extension may be ignored, certainly at the lower temperatures where, because of the high catenary force, fastening pull-out is more likely.

Fire attack from below the ceiling

When the lower face is exposed to fire each panel bows downwards. Delamination of the fire exposed face occurs when the strength of the adhesive layer is lost. The flexural strength of the panel assembly then approaches zero and collapse will occur unless one or both faces are restrained horizontally at the panel ends so that they become catenaries.

If only the lower face is horizontally restrained the catenary force in that face is a maximum because the dead load of the whole panel (upper face, core and lower face) has to be carried by the lower face and its fastenings to the support structure. The catenary force can be beneficially shared between both faces, though not equally, if both faces are horizontally restrained.

Fire attack from above the ceiling

When the upper face is exposed to fire each panel initially bows upwards. Delamination of the fire exposed face occurs when the strength of the adhesive layer is lost. The flexural strength of the assembly then approaches zero and collapse will occur unless at least the lower face is restrained horizontally at its ends so that it becomes a catenary

If only the lower face is horizontally restrained the catenary force in that face is a maximum because the dead load of the whole panel (upper face, core and lower face) has to be carried by the lower face and its fastenings to the support structure. The catenary force can be beneficially shared between both faces if both faces are horizontally restrained. It should be noted that the catenary force in the lower face will be high because there is no beneficial sag in the lower face in the absence of a temperature rise in the lower face. However the transfer of the dead load of the upper facing and core onto the lower face will cause some beneficial sag due to in-plane flexibility of the whole panel assembly e.g. by dragging together the panel end support structure, by slippage of the fastenings and/or elongation of the fastening holes in the panel facing. The estimate of inward end movement requires professional structural engineering judgement.

Of the two conditions, i.e. fire exposure from above or fire exposure from below, fire exposure from above causes the largest catenary forces. Note also that fire from above may be unseen by people, e.g. fire fighters, below the ceiling and collapse could present a life risk. The realistic assessment of catenary force and the use of properly designed and tested fastenings is therefore of great importance.

LOADING AND MATERIALS DATA FOR ELEVATED-TEMPERATURE CALCULATIONS

The dead load of the facing and core can be calculated from information on the volume and density of the construction materials. The density of steel sheet can be assumed to be 7850 kg/m^3. The density of other metal sheets can be obtained from national standards. The density of the core material at elevated temperature should be assumed to be the density at room temperature unless a) there are appropriate data available on the time-dependant change in density due to the effects of fire exposure e.g. due to charring or significant reduction in moisture content or b) the core material is consumed in the heating process, as with expanded polystyrene foam.

The reduction in strength properties of steel at elevated temperature may be assumed to vary according to the relevant national standard e.g. in the United Kingdom by reference to BS 5950: Part 8: 1980 which gives strength reduction factors for hot rolled steel and cold formed steel at different temperatures. Alternatively, information in the structural Eurocodes could be used, for instance, Eurocode 3: Design of steel structures, Part 1.2 General rules: Structural fire design (DD ENV 1993-1-2). Strength reduction factors for other metals may also be obtained from national standards or laboratory tests.

The weakest point in a panel assembly acting as a catenary is likely to be at the fastenings used to attach the panel ends to the supporting structure. Resistance to collapse is achieved by providing an adequate number of robust fastenings (e.g. stainless steel self-tapping screws) of known strength characteristics so that the catenary force is resisted. Information on elevated temperature performance of fastenings insitu may be obtained from the panel manufacturer’s test data

CONCLUSIONS

Sandwich panels have many advantages. Care is needed however to ensure that premature instability does not occur in a fire. Ways of retaining panel stability have been described and a theory developed for ceiling panels. Assessment of panel stability forms a part of the overall fire risk assessment for a building. The panel support structure must have at least the same fire resistance as the panel assembly.

Making calculations of the structural behaviour of a sandwich panel in fire is only necessary if the panel is to be used in an application which involves a span which is greater than the span tested in the fire-resistance test. Most fire resistance test furnaces adopt a span of approximately 4.5m whereas structural sandwich panel ceiling spans can reach 12m.

Without a calculation of the catenary forces acting on a ceiling sandwich panel it is possible for the fasteners at the ends of the panels to fail allowing collapse at an early stage in a fire. A simple calculation method has been proposed which ignores elastic extension of the face and enables the catenary force to be calculated for fire attack from above or below the ceiling.

To reduce the magnitude of catenary force developed when fire attack is from below the ceiling, it is advantageous to use an adhesive for bonding the core to the facing which weakens at a high a temperature as possible consistent with economy and panel production method.

Fire attack from above the ceiling will lead to the development of large and perhaps unsustainable catenary forces unless there is sufficient in-plane flexibility in the total ceiling assembly to allow some sagging of the lower facing. This requires an assessment using professional structural engineering judgement.

REFERENCES

1 Cooke G M E, When are sandwich panels safe in fire?- Part 1 An overview, *Fire Engineers Journal*, July 1998, pp 37 - 41

2 Cooke G M E, When are sandwich panels safe in fire?- Part 2 Avoiding collapse, *Fire Engineers Journal*, Sept 1998, pp 25 - 33

3 Cooke G M E, When are sandwich panels safe in fire?- Part 3 Fire scenarios, fire tests and checklist, *Fire Engineers Journal*, January 1999, pp 18 - 25

4 Cooke G M E, The behaviour of sandwich panels exposed to fire, *Building Engineer*, July 1997, p 14-29

5 Code of practice for fire protection in the food and drink industry, pub Loss Prevention Council, London, 1999

6 Guidelines for the design, specification, construction, maintenance and fire management of insulated envelopes for temperature controlled environments, pub The International Association of Cold Storage Contractors (European Division), 1999

7 Lightweight sandwich construction (edited J M Davies), published Blackwell Science, Oxford, UK, 2001

8 Cooke G M E, Sandwich panels for external cladding – fire safety issues and implications for the risk assessment process. Published by Eurisol, UK, November 2000, pp 60

9 Cooke G M E**,** Design fires for testing combustible-cored sandwich panels, Interflam 2001, International fire science and engineering conference held in Edinburgh on 17-19 Sept 2001, Pub Interscience Communications Ltd, London, 2001, pp 1189-1195

ANNEXES

Annex A

The classical text book equation for the horizontal restraint force H needed to support a catenary (cable-like structure) of span L carrying a uniformly distributed load w per unit length having a mid-span deflection D, Figure A1, is simply derived in the following way. For equilibrium $\sum M = 0$ where M = moment. Taking moments about point A for the right hand half of the catenary,

$$w\frac{L}{2}\frac{L}{2} = Hd + w\frac{L}{2}\frac{L}{4}$$ from which

$$H = \frac{wL^2}{8D} \qquad \text{(A1)}$$

Note that H becomes infinite as D becomes small

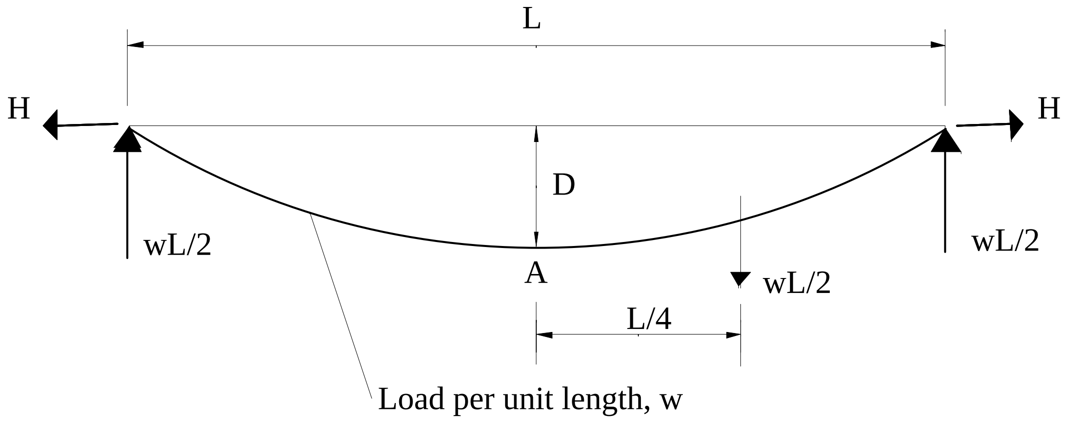

Figure A1 Catenary force diagram

Annex B

The equation derived below relates the axial shortening Δ_L to the mid-span deflection Δ_N for a flexible member when the member bows into a circular arc. Consider an initially straight member AB of length L, Figure B1, in which end A is position fixed. The member is slender so that the application of an axial compressive force P at end B causes negligible elastic compressive strain in the material but causes it to bow into a circular arc ACD.
From Figure B1

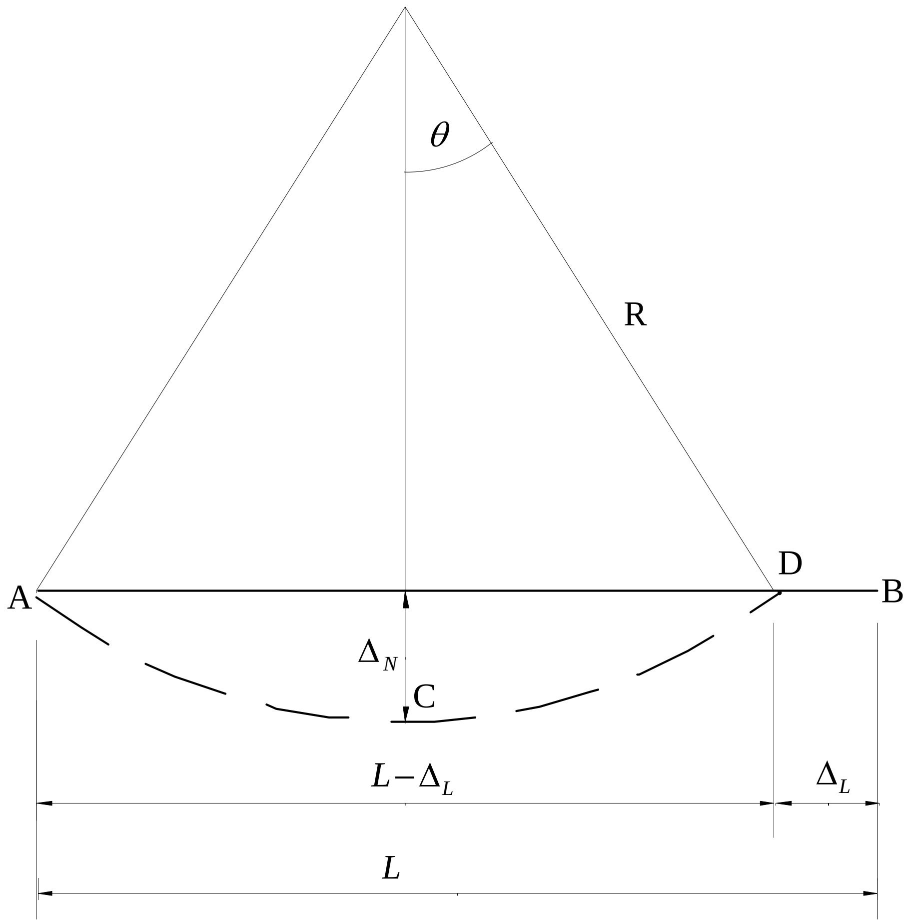

Figure B1 Geometry of bowed member

$$L = 2R\theta \tag{B1}$$

$$\frac{L-\Delta_L}{2} = RSin\theta \tag{B2}$$

$$R - \Delta_N = RCos\theta \tag{B3}$$

From Equations (B1) and (B2)

$$\Delta_L = 2R(\theta - Sin\theta) \tag{B4}$$

Substituting for R from (B2) in (B4)

$$\Delta_L = (L-\Delta_L)\frac{(\theta - Sin\theta)}{Sin\theta} \tag{B5}$$

By Maclaurin's series, $Sin\theta = \theta - \theta^3/3! + \theta^5/5! \approx \theta - \theta^3/3!$

Substituting in Equation (B5)

$$\Delta_L = (L - \Delta_L)\frac{\theta^2}{6} \qquad \text{(B6)}$$

From Equation (B3)

$\Delta_N = R(1 - Cos\theta)$ and substituting for R from Equation (B2) gives:

$\Delta_N = (L - \Delta_L)\dfrac{(1 - Cos\theta)}{2Sin\theta}$ and as $Cos\theta \approx 1 - \theta^2 / 2!$

$$\Delta_N = (L - \Delta_L)\frac{\theta}{4} \qquad \text{(B7)}$$

Form Equation (B6)

$\theta = \sqrt{\dfrac{6\Delta_L}{L - \Delta_L}}$ and substituting for θ in Equation (B7) gives:

$\Delta_N = \dfrac{(L - \Delta_L)}{4}\sqrt{\dfrac{6\Delta_L}{L - \Delta_L}}$ and ignoring 2nd order terms

$$\Delta_N = \sqrt{0.375 L\Delta_L} \qquad \text{(B8)}$$

If, instead of pushing end B of the slender member in by an amount Δ_L, the member is heated through a temperature T, the longitudinal expansion will be αLT where α is the coefficient of linear thermal expansion. If both ends of the member are position fixed before heating, the member will bow due to the expansion. In this case $\Delta_L = \alpha LT$ and substituting in Equation (B8) gives:

$$\Delta_N = L\sqrt{0.375\alpha T} \qquad \text{(B9)}$$

THE FOLLOWING THEORY DOES NOT FORM PART OF THE ABOVE PAPER. IT IS APPPENDED HERE SO AS TO ENABLE THOSE WITH AN INTEREST IN ROTATIONAL AND LATERAL DISPLACEMENTS DUE TO THERMAL BOWING TO HAVE ACCESS TO A SIMPLE THEORY IN A SPECIFIC AND GENERAL FORM.

Theory of unrestrained thermal bowing of a member

Synopsis

Here a simple theory of thermal bowing is developed based solely on geometry. It covers the case of a) an unrestrained simply supported beam and b) an unrestrained fixed-base cantilever, both having a linear temperature profile across the section. The theory provides linear displacements in the direction of heat flow, and rotational displacements at the ends and at intermediate positions along the member. The theory provides these data for a member in which the linear temperature profile across the section varies, and does not vary, along the length of the member. Some example calculations are given.

Assumptions

1 the coefficient of linear thermal expansion α does not vary throughout the member (i.e. the material is homogeneous) and is independent of temperature. For structural steels α is taken to be 14 x10^{-6}/^{o}C at elevated temperatures

2 The variation of temperature across the section in the direction of heat flow is linear

3 there is no variation of temperature in the section normal to the direction of heat flow

4 plane sections remain plane so that strain is proportional to the distance from the neutral axis

5 The section is free of internal stresses. It would not be if the temperature distribution was non-linear or if the member was subjected to external loads

Displacements of a non-loaded simply supported member having a linear temperature profile across its depth which does not vary with length.

Consider a simply supported member of length L and depth d, Figure A (3.1 in theses). It is subjected to heating on the upper face such that the temperature of the face is at a constant temperature throughout its length and the lower face is also at a constant temperature throughout its length, with the temperature varying linearly between the faces by an amount T_1. No external loads are applied. The member bows upwards in a circular arc of radius R and each end rotates through an angle θ. It remains free of internal stresses. For an element length dx, the expansion of the uppermost fibre is:

$$de = \alpha \frac{T_1}{2} dx$$

The angular change in element dx is:

$$d\theta = \frac{de}{h} = \alpha \frac{T_1}{2h} dx \qquad (1)$$

Integrating Equation (2) gives:

$$\theta = \int_0^{L/2} d\theta = \frac{\alpha T_1}{2h}\int_0^{L/2} dx = \frac{\alpha T_1}{2h} \frac{L}{2}$$

$$\theta = \frac{\alpha T_1 L}{2d} \qquad (2)$$

It is now necessary to express θ in terms of central displacement Δ. Using the properties of similar triangles it follows from Figure 2 (3.2 in theses) that CB/AC = ED/EB. For small θ, CB = L/4 so that:

$$\frac{L/4}{R} = \frac{\Delta}{L/2} \text{ from which:}$$

$$R = \frac{L^2}{8\Delta} \tag{3}$$

Ignoring Δ which is small compared with R,

$\theta = \tan^{-1}\frac{L/2}{R}$ from which, for small angles

$$\theta = \frac{L}{2R} \tag{4}$$

Substituting R from Equation (3) in Equation (4) gives

$\theta = \frac{4\Delta}{L}$ and substituting in Equation (2) gives:

$$\theta = \frac{\alpha T_1 L}{2d} \tag{5}$$

So that $\theta = \frac{4\Delta}{L}$

And therefore $$\Delta = \frac{\alpha T_1 L^2}{8d} \tag{6}$$

Displacements of a non-loaded, cantilevered member having a linear temperature profile across the section which does not vary with length

A cantilever, such as a column or wall, of height H and thickness d has a linear temperature profile caused by a temperature difference across the two faces of T_1. The member, if fixed at its base and free of external forces, will bow into a circular arc, Figure 3 (Fig 3.3 theses).

Consider the displacements of a small element of length dx. The thermal expansion of one face relative to the other is de and this causes an elemental rotation dθ. From Figure 3:

$$d\theta = \frac{de}{d} = \frac{\alpha T_1 dx}{d} \tag{7}$$

$$\Delta_H = \int_0^H x.d\theta \tag{8}$$

Substituting Equation (7) in (8) gives:

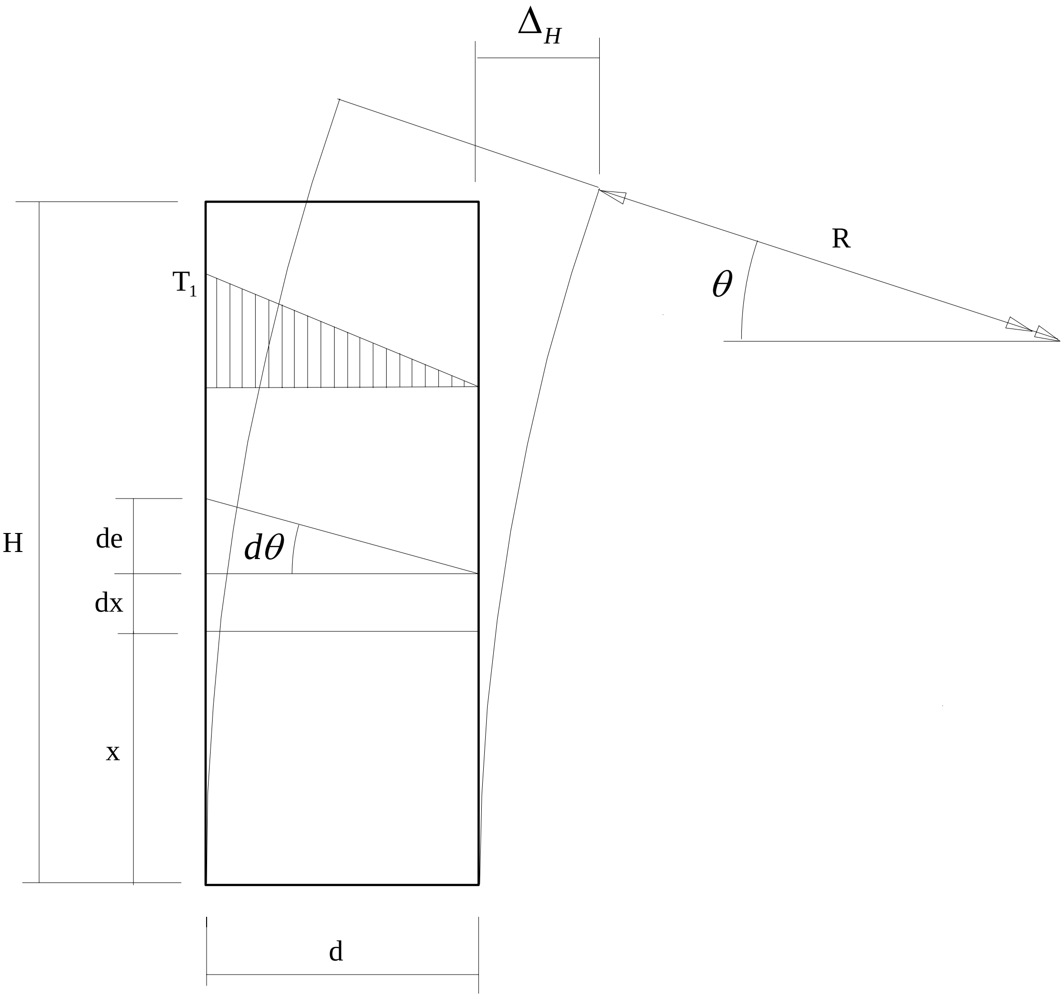

Figure 3 Thermal bowing of cantilever member having a linear temperature gradient across the depth

$$\Delta_H = \int_0^H x.\frac{\alpha T_1 dx}{d} = \frac{\alpha T_1}{d}\int_0^H x dx = \frac{\alpha T_1}{d}\left[\frac{x^2}{2}\right]_0^H$$

$$\Delta_H = \frac{\alpha T_1 H^2}{2d} \qquad (9)$$

Displacements of a non-loaded, simply supported member comprising four finite lengths, each having a different linear temperature profile across the section

Consider a member of total length L, Figure 4 (3.4 in thesis). Expressions will be developed for (i) rotations at the ends and at intermediate points, and (ii) vertical displacement at any point along the member.

Using the well-known relationship $\frac{d^2y}{dx^2} = \frac{1}{R}$ and as L = Rθ when θ is small:

$\frac{d^2y}{dx^2} = \frac{\theta}{L}$ and from Equation (2):

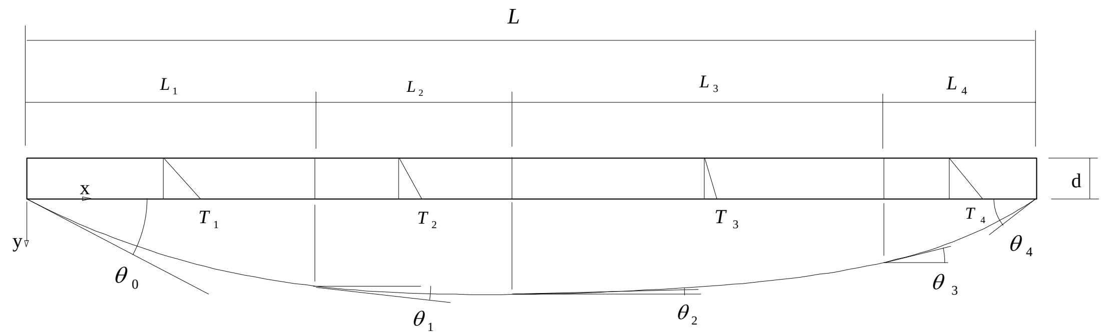

Figure 4 Simply supported member having linear temperature profiles across the depth of section which vary with length

$\frac{d^2y}{dx^2} = \frac{\alpha T_1}{d}$ and integrating gives $\frac{dy}{dx} = A + \frac{\alpha}{2d}\int T_1 dx$ where A is the constant of integration which equals θ_0 since dy/dx = θ_0 when x = 0. So that:

$$\frac{dy}{dx} = \theta_0 = \frac{\alpha}{d}\int T_1 dx \qquad (10)$$

$$\frac{dy}{dx} = \theta_0 + \frac{\alpha}{d} T_1 x \text{ for } x \le L_1$$

$$\frac{dy}{dx} = \theta_0 + \frac{\alpha}{d}\left[T_1 L + T_2(x - L_1)\right]$$

$$\frac{dy}{dx} = \theta_0 + \frac{\alpha}{d}\left[T_1 x + (T_2 - T_1)(x - L_1)\right] \text{ for } L_1 \le x \le (L_1 + L_2)$$

$$\frac{dy}{dx} = \theta_0 + \frac{\alpha}{d}\left[T_1 x + (T_2 - T_1)(x - L_1) + (T_3 - T_2)(x - L_1 - L_2)\right] \text{ for } L_1 + L_2 \le x \le L_1 + L_2 + L_3$$

Hence by further integration

$$y = \theta_0 x + \frac{\alpha}{2d} T_1 x \quad \text{for } x \le L$$

$$y = \theta_0 x = \frac{\alpha}{2d}\left[T_1 x^2 + (T_2 - T_1)(x - L_1)^2\right] \text{ for } L_1 \le x \le L_1 + L_2 \qquad (11)$$

$y = 0$ when $x = L = L_1 + L_2 + L_3 + L_4$ hence:

$$y = \theta_0$$

$$(L_1 + L_2 + L_3 + L_4) + \frac{\alpha}{2d}\left[T_1(L_1 + L_2 + L_3 + L_4)^2 + (T_2 - T_1)(L_2 + L_3 + L_4)^2 + (T_4 - T_3)L_4^2 = 0\right]$$ Tra

nsposing gives:

$$\theta_0 = -\frac{\alpha}{2d(L_1 + L_2 + L_3 + L_4)}\left[T_1(L_1 + L_2 + L_3 + L_4)^2 + (T_2 - T_1)(L_2 + L_3 + L_4)^2 + (T_3 - T_2)(L_3 + L_4)^2 + (T_4 - T_3)L_4^2\right] \qquad (12)$$

When $x = L_1$, $\theta_1 = \frac{dy}{dx} = \theta_0 + \frac{\alpha}{d} T_1 L_1$ Similarly:

$$\theta_2 = \theta_0 + \frac{\alpha}{d}\left(T_1L_1 + T_2L_2\right)$$

$$\theta_3 = \theta_0 + \frac{\alpha}{d}\left(T_1L_1 + T_2L_2 + T_3L_3\right) \text{ and}$$

$$\theta_4 = \theta_0 + \frac{\alpha}{d}\left(T_1L_1 + T_2L_2 + T_3L_3 + T_4L_4\right) \qquad (13)$$

Worked example of calculation of member end rotation

L_1=1000mm, L_2 = 1500mm, L_3 = 1000mm, L4 = 500mm, T_1=200^{o}C, T_2 = 250^{o}C, T_3=160^{o}C, T_4=100^{o}C, d=50mm, α = 0.000014/^{o}C for steel. Find the end rotations.

From Equation (12):

$$\theta_0 = -\frac{0.000014}{2\times 50\times 4000}\left[200\times 4000^2 + 50\times 3000^2 - 90\times 1500^2 - 60\times 500^2\right]$$

$\theta_0 = -$ 0.12013 radians

From equation (13):

$$\theta_4 = \theta_0 + \frac{0.000014}{50}\left(785000\right) = -0.12013 + 0.2198 = 0.09967 \text{ Radians}$$

Total rotation = $\theta_0 + \theta_4$ = 0.21977 radians. This can be checked against the total rotation using an average temperature difference, so:

Average temperature difference per unit length is given by

$$\frac{L_1T_1 + L_2T_2 + L_3T_3 + L_4T_4}{L_1 + L_2 + L_3 + L_4} = \frac{1000\times 200 + 1500\times 250 + 1000\times 160 + 500\times 100}{4000}$$

= 196.25^{o}C

As the total rotation is $2\theta_0$, the total rotation, using Equation (5), is

$$\frac{2\alpha TL}{2d} = \frac{0.000014\times 196.25\times 4000}{50} = 0.2198$$ Radians which agrees wi th the answer (0.21977) from the rigorous method above.

The rigorous method above can be generalized. From inspection of Equation (12) it is clear that any term in the square brackets may be written in the general form:

$$\left(T_i - T_{i-1}\right)\left\{L - \sum_i^{i-1} L_i\right\}^2$$ So that;

$$\theta_0 = -\frac{\alpha}{2dL}\sum\left(T_i - T_{i-1}\right)\left\{L - \sum_i^{i-1} L_i\right\}^2 \qquad (14)$$

Equation (14) and the others above have been validated against experimental work by the author (Cooke G M E, The structural response of steel I-section members subjected to elevated temperature gradients across the section, PhD thesis, City University, London, Sept 1987, pp 450).

Application constraints

i) The above theory assumes the coefficient of thermal expansion does not vary with temperature It does vary (ie from 12 $x10^{-6}/^oC$ at room temperature to $14x10^{-6}/^oC$ at temperatures around 600^oC for structural steel) but it is common to use a value of $14x10^{-6}/^oC$ in fire engineering calculations
ii) The theory only gives rotations and deflections before onset of plastic deformations and before elastic modulus is greatly affected by temperature. The theory is nonetheless very useful up to this point
iii) In practice temperature profiles are rarely linear but the theory still works if a best-fit linear equivalent of the curvilinear temperature profile is used.